$$\eta = \eta_0 f(I), \quad I = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2, \quad f > 0$$

As usual, we introduce the stream function Ψ by putting $u = \partial\Psi / \partial y$ and $v = -\partial\Psi / \partial x$ and eliminate the pressure p from the first two equations of (1.1). This yields the following quasilinear, fourth order equation for

$$L(\Psi) \equiv a_1\left(\frac{\partial^4\Psi}{\partial x^4} + \frac{\partial^4\Psi}{\partial y^4}\right) + 2a_2\frac{\partial^4\Psi}{\partial x^2\,\partial y^2} + 4a_3\left(\frac{\partial^4\Psi}{\partial x\,\partial y^3} - \frac{\partial^4\Psi}{\partial x^3\,\partial y}\right) = H(\Psi) \quad (1.2)$$

where $H(\Psi)$ is a nonlinear third order operator and

$$a_1 = \eta_0 (f + 2\varepsilon^2 f'), \qquad a_2 = \eta_0 (f + 4\varepsilon_1^2 f' - 2\varepsilon^2 f'), \qquad a_3 = 2\eta_0\varepsilon\varepsilon_1 f'$$

$$\varepsilon = \partial u / \partial y + \partial v / \partial x = \partial^2\Psi / \partial y^2 - \partial^2\Psi / \partial x^2, \qquad \varepsilon_1 = 2\partial u / \partial x = 2\partial^2\Psi / \partial x\,\partial y$$

It can be shown by usual methods, that the presence of the real characteristics of (1.2) on the xy-plane depends on the form of the function $f(I)$. When the inequality $f+2If' > 0$ holds, then the real characteristics are absent. In particular, they are absent in the case of the normal type hydrodynamics when $f' = 0$, and for any model with increasing viscosity, when $f' > 0$.

If, on the other hand

$$f + 2If' < 0 \quad (1.3)$$

then four families of real characteristics $y = y(x)$ exist, and are given by

$$\left(\frac{dy}{dx}\right)_{1,2,3,4} = k_\pm \pm \sqrt{k_\pm^2 + 1}, \qquad k_\pm = \frac{4\varepsilon\varepsilon_1 f' \mp \sqrt{-f^2 - 2Iff'}}{2(f + 2\varepsilon^2 f')} \quad (1.4)$$

Thus, when (1.3) holds, (1.2) becomes hyperbolic. Condition (1.3) implies that the viscosity should decrease with increasing rate of strain and, that it should do so sufficiently fast to ensure that $df / d\sqrt{I} < -f / \sqrt{I}$. It can easily be seen that the inequality (1.3) corresponds to a segment on the curve $\sqrt{J} = F(\sqrt{I})$ where $J = 4\eta_0 f^2 I$ is the second invariant of the viscous stress tensor, on which the stresses decrease with increasing rate of strain and, that the region of hyperbolicity of (1.2) corresponds to this segment. An interesting fact emerges, namely, thay in the flow of a Newtonian fluid whose viscosity is temperature dependent, the instability is indicated by the appearance of a similar decreasing segment on the curve connecting, say, the friction on the wall with its rate of flow [1]. When this happens, then the system of equations does not assume real characteristics under any conditions.

Analogous results are obtained in a number of other problems, e. g. when considering a fully developed flow in a pipe, we obtain the following quasilinear, second order equation for the longitudinal velocity

$$N(u) \equiv \left[f + 2f'\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial y^2} + 4f'\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\,\partial z} +$$

$$+ \left[f + 2f'\left(\frac{\partial u}{\partial z}\right)^2\right]\frac{\partial^2 u}{\partial z^2} = -\frac{P}{\eta_0}, \qquad P = -\frac{\partial p}{\partial x} = \text{const} \quad (1.5)$$

which also becomes hyperbolic when the condition (1.3) holds.

Investigation of real flows such as Couette or Poiseville flows indicates that Eqs. (1.2) or (1.5) indeed have solutions satisfying the appropriate boundary conditions, provided that the inequality (1.3) holds for a part (or sometimes for the whole) of the investigated region of flow.

It can naturally be assumed that the behavior of the fluid is Newtonian, when the rates of strain are small. From this it follows that the regions in which (1.3) holds, can only appear in the flow when the pressure gradients are sufficiently large (flows in pipes) or, when the boundaries move with high velocities (Couette flows). We can introduce the concept of the critical shear rate D_* at which J reaches its first maximum and $f + 2If'$ changes its sign from plus to minus, and thus obtain a class of media with the critical shear rate.

Thus a possibility exists, in principle, that a "hyperbolic" mode of nonlinear viscous flow can be assumed by the media with the critical shear rate. Under certain boundary conditions the inequality $\sqrt{I} > D_*$ will hold for these media in some parts of the flow. It should however be stressed, that at present we are not discussing the existence of real flows, but the implications which can be derived from the initial system of equations.

We can further consider the corresponding unsteady problems. We shall of course find, that in this case (1.2) and (1.5) are replaced by

$$\rho\partial\Delta\Psi / \partial t = L(\Psi) - H(\Psi), \qquad \rho\partial u / \partial t = \eta_0 N(u) + P \tag{1.6}$$

where the right-hand sides are the operators which change their character when $f + 2If'$ passes through zero.

We note that equations analogous to (1.5) are also encountered in other problems, e.g. in the electrodynamics of the media with nonuniform conductivity [2 and 3] and, that the existence of an inequality equivalent to (1.3) has been confirmed there without a doubt.

2. We shall now consider the flow of an elastoviscous fluid with constant properties, e.g. an unsteady flow of such a fluid in a plane channel $0 < y < \delta$ with impermeable walls, assuming that all parameters depend on the time t and the transverse coordinate y. For simplicity, we shall use the Oldroyd model with two constants

$$T_{ij} = -p\delta_{ij} + \tau_{ij}, \quad \tau_{ij} + \lambda\tau_{ij}^{\bullet} = 2\eta e_{ij}, \quad \tau_{ij}^{\bullet} \equiv \frac{\partial\tau_{ij}}{\partial t} + v_k\tau_{ij,k} - v_{i,k}\tau_{kj} - v_{j,k}\tau_{ik}$$

assuming a plane state of stress. Then the velocity and the stress tensor components will be given by

$$\rho\frac{\partial u}{\partial t} = P(t) + \frac{\partial\tau_{xy}}{\partial y}, \qquad \frac{\partial p}{\partial y} = \frac{\partial\tau_{yy}}{\partial y} \tag{2.1}$$

$$\tau_{xx} + \lambda\left(\frac{\partial\tau_{xx}}{\partial t} - 2\tau_{xy}\frac{\partial u}{\partial y}\right) = 0, \qquad \tau_{yy} + \lambda\frac{\partial\tau_{yy}}{\partial t} = 0$$
$$\tau_{xy} + \lambda\left(\frac{\partial\tau_{xy}}{\partial t} - \tau_{yy}\frac{\partial u}{\partial y}\right) = \eta\frac{\partial u}{\partial y} \tag{2.2}$$

Second equation of (2.1) and the first equation of (2.2) can be used to obtain the transverse distribution of pressure and the magnitude τ_{xx} in terms of known u, τ_{xy} and τ_{yy}. First equation of (2.1) and third equation of (2.2) define u and, after τ_{yy} has been obtained from the second equation of (2.2), also τ_{xy}.

We shall asume that at the initial instant $\tau_{yy} = \tau_0 = \text{const}$; then a continuous solution of the second equation of (2.2) will have the form $\tau_{yy} = \tau_0 \exp(-t/\lambda)$, (2.1) and the last equation of (2.2) will yield

$$\frac{\partial^2 u}{\partial t^2} - \frac{\eta/\lambda + \tau_0 e^{-t/\lambda}}{\rho}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t} = \frac{1}{\lambda\rho}[P(t) + \lambda P'(t)] \equiv Q(t) \tag{2.3}$$

Let us now suppose that $Q(t)$ is given and that u satisfies

$$u(0, y) = u_0(y), \quad \partial u(t, y)/\partial t\,|_{t=0} = u_1(y), \quad u(t, 0) = u(t, \delta) = 0 \tag{2.4}$$

Then the solution $u(t, y)$ can be written as

$$u(t, y) = \sum_{k=1}^{\infty} u_k(t) \sin \alpha_k y, \qquad \alpha_k = \frac{k\pi}{\delta}$$

the functions $u_k(t)$ will satisfy

$$u_k'' + \frac{1}{\lambda} u_k' + \frac{1}{\rho} (\eta / \lambda + \tau_0 e^{-t/\lambda}) \alpha_k^2 u_k = \frac{2}{k} [1 + (-1)^{k-1}] Q \tag{2.5}$$

Assuming that $\tau_0 \neq 0$ (the case $\tau_0 = 0$ has been studied by various authors and the solution obtained in terms of elementary functions) and using the initial conditions (2.4) for $u_k(t)$, we obtain the following Formulas:

$$u_k(t) = \exp\left(-\frac{t}{2\lambda}\right)\Big[A_k J_{\nu_k}(r_k(t)) + B_k Y_{\nu_k}(r_k(t)) +$$

$$+ \frac{\pi r_k(0)}{2} \int_0^t \{J_{\nu_k}(r_k(t)) Y_{\nu_k}(r_k(t')) - Y_{\nu_k}(r_k(t)) J_{\nu_k}(r_k(t'))\} Q(t')\, dt'\Big] \tag{2.6}$$

$$r_k(t) = 2\lambda\alpha_k \left(\frac{\tau_0}{\rho}\right)^{1/2} \exp\frac{-t}{2\lambda}, \qquad \nu_k = \left(1 - \frac{4\alpha_k^2 \lambda\eta}{\rho}\right)^{1/2} \quad (\tau_0 > 0)$$

$$u_k(t) = \exp\left(-\frac{t}{2\lambda}\right)\Big[C_k I_{\nu_k}(s_k(t)) + D_k K_{\nu_k}(s_k(t)) -$$

$$- s_k(0) \int_0^t \{I_{\nu_k}(s_k(t)) K_{\nu_k}(s_k(t')) - K_{\nu_k}(s_k(t)) I_{\nu_k}(s_k(t'))\} Q(t')\, dt'\Big] \tag{2.7}$$

$$s_k(t) = 2\lambda\alpha_k \left(-\frac{\tau_0}{\rho}\right)^{1/2} \exp\frac{-t}{2\lambda} \qquad (\tau_0 < 0)$$

where the constants A_k, B_k and C_k, D_k are given by the following systems of linear equations

$$J_{\nu_k}(r_k(0)) A_k + Y_{\nu_k}(r_k(0)) B_k = E_k \tag{2.8}$$

$$J_{\nu_k}'(r_k(0)) A_k + Y_{\nu_k}'(r_k(0)) B_k = -\frac{E_k + 2\lambda F_k}{r_k(0)} \quad \text{when } \tau_0 > 0$$

$$I_{\nu_k}(s_k(0)) C_k + K_{\nu_k}(s_k(0)) D_k = E_k$$

$$I_{\nu_k}'(s_k(0)) C_k + K_{\nu_k}'(s_k(0)) D_k = -\frac{E_k + 2\lambda F_k}{s_k(0)} \quad \text{when } \tau_0 < 0 \tag{2.9}$$

The quantities E_k and F_k appearing in the right-hand sides of (2.8) and (2.9) represent the Fourier coefficients of the functions $u_0(y)$ and $u_1(y)$. The determinants of the systems (2.8) and (2.9) are, obviously, not equal to zero.

Returning to Eq. (2.3) we easily see that it is hyperbolic (just as in the case $\tau_0 = 0$) when the inequality $\tau_0 \exp(-t/\lambda) + \eta/\lambda > 0$ holds. Equation (2.3) is hyperbolic for all values of t, provided that $\tau_0 > -\eta/\lambda$. Otherwise, when the inequality $\tau_0 < -\eta/\lambda$ holds, then (2.3) belongs to the mixed type and the time intervals $(0, t_*)$ and (t_*, ∞) where $t_* = -\lambda \ln(-\eta/\lambda\tau_0)$ corresponds to elliptic and hyperbolic regions of the equation. From the above we can easily infer that a formal solution of the problem (2.3) and (2.4) can always be obtained irrespective of the type of (2.3), but when (2.3) is of the mixed type, then there is no physical counterpart to the region of ellipticity.

3. Adamar type of examples can easily be constructed for (2.3) in its given form and for the linearized equations (1.6). These examples show that, when $\tau_0 < -\eta/\lambda$ in (2.3) or, when the operator in the right-hand side of (1.6) is hyperbolic, then the initial value

problems for these equations are stated incorrectly or, in other words, with the given initial conditions these equations are nonevolutionary.

So, the general conclusion which follows is, that the hydrodynamic equations of a non-Newtonian fluid may become nonevolutionary, the latter fact depending on the conditions called for by the particular rheological model (similar conclusion, though based on different assumptions, were obtained in [5], where the propagation of small perturbations in an elastic medium with finite deformations was investigated).

The lack of evolutionarity can only be detected in nonsteady flows; we see however from Section 1, that the investigation of steady flows furnishes some reasons for it [6].

The lack of evolutionarity of equations, becoming manifest under certain conditions means that under these conditions an instantaneous growth of small perturbations takes place. In the real systems, instability may develop only at some finite rate and the perturbation amplitudes cannot increase without bounds. The physical mechanism present, always insure that the above restrictions hold. Therefore the rheological models generating nonevolutionary equations should be improved by introducing the terms based on well known theories [4], which would ensure that the initial value problem is stated correctly. This, of course, does not eliminate the possibility that the improvement of the model leads to noticeable alteration of equations even within the "region of evolutionarity".

The supposition that the general principles of constructing the models of continuous media should include the necessary condition of the evolutionarity of equations, seems therefore feasible.

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BIBLIOGRAPHY

1. Joseph, D. D., Variable viscosity effects on the flow and stability of the flow in channels and pipes. Phys. Fluids, Vol. 7, №11, 1964.
2. Emets, Iu. P., The hodograph method in electrodynamics of continuous non-linearly conducting media. PMM Vol. 31, №6, 1967.
3. Oliver, D. A. and Mitchner, M., Nonuniform electrical conduction in MHD channels. AIAA Journal, Vol. 5, №8, 1967.
4. Sedov, L. I., Introduction to the Mechanics of Continuous Media. M., Fizmatgiz, 1962.
5. Hayes, M. and Rivlin, R. S., Propagation of a plane wave in an isotropic elastic material subjected to pure homogeneous deformation. Arch. for Rat. Mech. and Analysis, Vol. 8, №1, 1961.
6. Kulikovskii, A. G. and Regirer, S. A., On stability and evolutionarity of the electric current distribution in a medium with nonlinear conductivity. PMM Vol. 32, №4, 1968.

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